Göteborg ITP 94-26
December 1994
hep-th/9501084



# Construction of BRST invariant states in $G/H$ WZNW models

Stephen Hwang[1]
and
Henric Rhedin[2]

Institute of Theoretical Physics
Chalmers University of Technology
and
Göteborg University

**Abstract**

We study the cohomology arising in the BRST formulation of G/H gauged WZNW models, i.e. in which the states of the gauged theory are projected out from the ungauged one by means of a BRST condition. We will derive for a general simple group $H$ with arbitrary level, conditions for which the cohomology is non-trivial. We show, by introducing a small perturbation due to Jantzen, in the highest weights of the representations, how states in the cohomology, "singlet pairs", arise from unphysical states, "Kugo-Ojima quartets", as the perturbation is set to zero. This will enable us to identify and construct states in the cohomology. The ghost numbers that will occur are $\pm p$, with $p$ uniquely determined by the representations of the algebras involved. Our con-

[1]tfesh@fy.chalmers.se
[2]hr@fy.chalmers.se

struction is given in terms of the current modes and relies on the explicit form of highest weight null-states given by Malikov, Feigen and Fuchs.

WZNW models are of great importance in the study of conformal field theories. In particular, the gauging of these models is an essential part to be able describe most conformal field theories. This gauging will lead to an effective action, which is BRST invariant [1] and apart from the ungauged WZNW models based on the Lie group $G$ it contains an auxiliary and non-unitary WZNW model based on the gauged subgroup $H$ of $G$, and a ghost sector. In [2] we investigated the implications of the BRST symmetry, when applied in an operator formulation of these models. There it was shown that under certain restrictions, the BRST approach gave a consistent operator formulation of the model and the only states that survived were ghost free and satisfied the usual highest weight conditions w.r.t. $\hat{h}$, the affine Lie algebra corresponding to $H$. The restrictions were on the choice of representations: integral representations of the ungauged WZNW model and a corresponding selection of representations of the auxiliary WZNW model.

In this and a subsequent paper [3], we will study a much more general set of representations, namely arbitrary highest weight representations for both the original and auxiliary sectors. The motivation for this is that the states in the cohomology for non-zero ghost numbers, which we will show are present for more general representations, may be of importance in the determination of the complete physical spectrum in a specific model. In particular, it is generally believed to be true for topological theories.

We will here concentrate on the construction of the states in the cohomology. In [3] a more detailed analysis of the cohomology will be presented. Some of the results derived there will be needed in the present work, although most of the analysis presented here will be self-contained. Our analysis will always be confined to the relative cohomology i.e. in which the $b$-ghosts corresponding to the Cartan generators are required to annihilate states. We will derive general conditions and consistency equations for the non-trivial states in the cohomology and then give a procedure for constructing such states.

Our construction relies basically on the results and techniques presented by us previously [2] and the construction of highest weight null-states given by Malikov, Feigen and Fuchs [4]. Crucial is also a trick due to Jantzen [5] to perturb representations. With this perturbation one will clearly see how the non-trivial states in the cohomology, *"singlet pairs"*, emerge from the set of trivial and non-invariant states, *"quartets"* in the terminology of Kugo and Ojima [6]. The ghost numbers $\pm q$ of the singlet pairs, will for fixed representations of the original and auxiliary sectors, have

one fixed value of $q \neq 0$. These values will be the only ones possible [3]. We have, however, been unable to determine the dimensionality of the cohomology. In the simple examples that we have checked, the dimensionality is one.

Our analysis will be for $G/H$ gaugings with a general simple group $H$ of arbitrary level. However, there are some limitations due to the lack of complete decomposability of representations of $\hat{g}$ w.r.t. $\hat{h}$. In two cases it is known that such limitations do not exist, namely if we consider integrable representations of $\hat{g}$ or if $G = H$. Thus our work will be completely general for these cases. In the generic case, however, one does not have complete decomposability. Then our analysis must be interpreted with some care. The implications of the lack of decomposability is that certain null-states of $\hat{h}$ will not be null-states w.r.t. $\hat{g}$. This in turn implies that states in the cohomology of $H/H$ models, which are based on these null-states of $\hat{h}$ will no longer be in the cohomology in the $G/H$ case. Since the presence of states at ghost numbers $\pm q$ will imply states of ghost numbers $\pm(q+1), \pm(q+2), \ldots$ (for other representations), infinitely many states may drop out.

In [7]-[9] some investigations of the cohomology using free field techniques were presented. In particular, in [7], conjecturing the cohomology of the Felder operator to be one-dimensional for exaclty one degree it was shown, quite generally, that this implied an analogous result for the BRST cohomology in the case of integrable representations, i.e. one-dimensional at exactly one ghost number and its negative. The results presented here and in [3] will confirm this result as far as the ghost numbers are concerned, but as stated above, we have no results concerning the dimensionality.

We start by giving a qualitative explanation of this note. One of the basic ingrediences is a theorem due to Kugo and Ojima [6], which by using the BRST symmetry classifies the states of an irreducible state-space with a well-defined and finite inner product. The theorem states that it is possible to choose a basis consisting of states of the three categories:

*singlets*: BRST non-trivial states $|S_0\rangle$ at ghost number zero, $\langle S_0|S_0\rangle \neq 0$

*singlet pairs*: BRST non-trivial states $|S_q\rangle, |\bar{S}_{-q}\rangle$ of opposite ghost number, $\langle \bar{S}_{-q}|S_q\rangle \neq 0$

*quartets*: BRST trivial states and their conjugates represented by the states $|S_{q-1}\rangle,\ |S_q\rangle,\ |\bar{S}_{-q}\rangle,\ |\bar{S}_{-q+1}\rangle$, where the index denotes the ghost number, and $\langle \bar{S}_{-q}|S_q\rangle \neq 0$, $\langle \bar{S}_{-q+1}|S_{q-1}\rangle \neq 0$, $\hat{Q}|S_{q-1}\rangle = |S_q\rangle$, and $\hat{Q}|\bar{S}_{-q}\rangle = |\bar{S}_{-q+1}\rangle$.

Here $\hat{Q}$ is the appropriate BRST operator.

The state space in our case consists of highest weight Verma modules over affine Lie algebras. If the Verma module is irreducible then one may show that only singlets and quartets appear [2]. For general weights on the weight lattice the Verma module is reducible, and singlet pairs will also be present. The basic idea behind our construction is to use a trick due to Jantzen [5], in which one adds a small perturbation to the highest weight $\lambda \to \lambda + \epsilon z$ where $z$ is an appropriate vector. For $0 < \epsilon \ll 1$ the Verma module is irreducible, so we will again only have singlets and quartets. In the limit $\epsilon \to 0$ the quartets $|S_{q-1}\rangle_\epsilon,\ |S_q\rangle_\epsilon,\ |\bar{S}_{-q}\rangle_\epsilon\ |\bar{S}_{-q+1}\rangle_\epsilon$ at non-zero ghost number may fall into one of the following categories: i) the quartet becomes an ordinary quartet of the reducible Verma module, ii) all four states in the quartet become null-states, iii) two states of the quartet turn into null-states and the remaining two become BRST non-trivial. For the third case, which is the interesting one, we find that the latter states form a singlet pair and these states are non-trivial in the cohomology. The idea is to identify the quartets which in the limit $\epsilon \to 0$ give singlet pairs, in particular to identify the states in the quartets that will become appropriate null-states. This will be one of the key results in our work. In particular, we will show that the states $|S_q\rangle_\epsilon$ of the quartet, in the relevant case, will have positive ghost numbers, while $|\bar{S}_{-q}\rangle_\epsilon$ will have negative ghost numbers i.e. that $\hat{Q}|S_{q-1}\rangle_\epsilon = \epsilon|S_q\rangle_\epsilon$ and $\hat{Q}|\bar{S}_{-q}\rangle = |\bar{S}_{-q+1}\rangle$ where $lim_{\epsilon\to 0}|S_{q-1}\rangle_\epsilon = |N_{q-1}\rangle$ and $lim_{\epsilon\to 0}|\bar{S}_{-q+1}\rangle_\epsilon = |\bar{N}_{-q+1}\rangle$ are null-states. We will also find that it is sufficient for our construction to know the highest weight null-states of the Verma module, and these are given by Malikov, Feigin and Fuchs [4].

We consider a Lie group $G$ valued WZNW model gauged with an anomaly free subgroup $H$. The resulting action is decomposable into three parts [10],[11], the original WZNW action, an auxiliary $H$ valued WZNW action, and a Fadeev-Popov ghost part. The different sectors have symmetry algebras of affine Lie type $\hat{g}_k$, $\hat{h}_{\tilde{k}}$, and $\hat{h}_{k_{gh}}$, respectively, of levels $k$, $\tilde{k} = -k - 2c_h$, and $k_{gh} = 2c_h$ where $c_h$ is the quadratic Casimir of the adjoint representation of the Lie algebra $h$. We take $k > -c_h$ since the case $k < -c_h$ may be considered by interchanging the original and auxiliary sectors. It is assumed that the state space decomposes into a $\hat{h}_k$ highest weight Verma module $M^h_\lambda$, an auxiliary $\hat{h}_{\tilde{k}}$ highest weight Verma module $\widetilde{M}^h_{\tilde{\lambda}}$ and a ghost module $\mathcal{F}$. Define a BRST operator on the space $M^h_\lambda \times \widetilde{M}^h_{\tilde{\lambda}} \times \mathcal{F} \equiv M^h_{\lambda,\tilde{\lambda}} \times \mathcal{F}$,

which is of the form [1]

$$Q = \oint \frac{dz}{2i\pi} \left[ : c_a(z)(J^a(z) + \tilde{J}^a(z)) : - \frac{i}{2} f^{ad}_{\ \ e} : c_a(z) c_d(z) b^e(z) : \right], \tag{1}$$

where $c_a(z)$ and $b^a(z)$ are ghost fields.

We will here study the relative cohomology $\hat{\mathrm{H}}^p(\hat{h}_k, \hat{h}_{\tilde{k}}; \ldots)$ which is the cohomology of the sub-complex defined such that all states on the subspace satisfy $b_0^i|\phi\rangle = 0$, $i = 1, ..., r_h$, where $r_h$ is the rank of $h$. It is then appropriate to consider the following decomposition of the BRST charge $Q = \hat{Q} + c_{0,i} J_0^{i,tot} + M_i b_0^i$. Here $J_0^{i,tot} = \{Q, b_0^i\}$ is the Cartan generator of the total current i.e. the sum of the currents of the original, auxiliary and ghost sectors. On the subspace $\hat{Q}$ is nilpotent, so it makes sense to use it as a BRST operator.

The Verma module is reducible if and only if it contains highest weight null vectors that are not proportional to the primary state $|0; \lambda\rangle$. We define the irreducible $\hat{h}$ module $L_\lambda^h$ as the $\hat{h}$ highest weight Verma module modulo the maximal proper Verma submodule the latter being the module containing all $\hat{h}$ Verma modules over highest weight null-states not proportional to $|0; \lambda\rangle$. States that are in the maximal proper submodule are called null-states. We define the module $L^h_{\lambda,\tilde{\lambda}} = L^h_\lambda \times \tilde{L}^h_{\tilde{\lambda}}$. It is the relative cohomology on this module which is our primary interest here. We start, however, by considering the relative cohomology of the full Verma module.

<u>THEOREM.</u> $\hat{\mathrm{H}}^p(\hat{h}_k, \hat{h}_{\tilde{k}}; M^h_{\lambda,\tilde{\lambda}})$ *vanishes for* $p < 0$.

<u>PROOF.</u> The theorem follows from an analogous treatment to the one presented in [2], so we will content ourselves with a brief sketch of the proof. We introduce a gradation of states which essentially counts excitations: $\sharp\tilde{J} + \sharp b - \sharp c$. The BRST equation may now be solved order by order in this gradation. The BRST charge decomposes as $d_0 + d_{-1}$, where $d_0$ when acting on a state with maximal degree $N$, gives states of maximal degree $N$ or lower. It is possible to define a homotopy operator $\kappa_0$ such that $(d_0\kappa_0 + \kappa_0 d_0)|\psi\rangle \propto |\psi\rangle$ to highest order and for positive degrees. Using this operation iteratively one eliminates higher orders in favour of states at lower order. This may be continued until the maximal degree is zero or lower. From the gradation one finds that negative ghost number states will always have positive degrees, and hence they must be BRST trivial. This gives the theorem. □

<u>COROLLARY 1.</u> *Let* $\widetilde{M}^h_{\tilde{\lambda}}$ *be irreducible then* $\hat{\mathrm{H}}^p(\hat{h}_k, \hat{h}_{\tilde{k}}; L^h_\lambda \times \widetilde{M}^h_{\tilde{\lambda}})$ *is zero for* $p \neq 0$, *and for* $p = 0$ *contains states of the form* $|0; \lambda\rangle|0; \tilde{\lambda}\rangle|0\rangle^+$. $|0; \lambda\rangle$ *and* $|0; \tilde{\lambda}\rangle$ *are highest weight primaries w.r.t.* $\hat{h}_k$ *and* $\hat{h}_{\tilde{k}}$, *respectively. The weights satisfy* $\lambda + \tilde{\lambda} + \rho = 0$,

*where $\rho$ is the sum of the positive roots of $h$. $|0\rangle^{+} = \prod_{\alpha\in\Delta_h^+} c_0^{\alpha}|0\rangle_{gh}$, where $|0\rangle_{gh}$ is the $Sl(2,R)$ invariant ghost vacuum and $\Delta_h^+$ is the set of positive roots of $h$.*

<u>PROOF.</u> This case was the one considered in [2]. The condition $\lambda+\tilde{\lambda}+\rho=0$ comes from the requirement $J_0^{tot,i}|\ldots\rangle = 0$. $\square$

<u>COROLLARY 2.</u> *Let $|\bar{S}\rangle \in \hat{\mathrm{H}}^p(\hat{h}_k, \hat{h}_{\tilde{k}}; L^h_{\lambda,\tilde{\lambda}})$ for $p<0$. Then on $M^h_{\lambda,\tilde{\lambda}}$, $\hat{Q}|\bar{S}\rangle = |\bar{N}\rangle$ for some non-zero $|\bar{N}\rangle \in M^h_{\lambda,\tilde{\lambda}}/L^h_{\lambda,\tilde{\lambda}}$*

<u>PROOF.</u> Assume the contrary i.e. $\hat{Q}|\bar{S}\rangle = 0$. Then by the theorem $|\bar{S}\rangle = \hat{Q}|\bar{S}'\rangle$. Now if $|\bar{S}\rangle \in L^h_{\lambda,\tilde{\lambda}}$ then $|\bar{S}'\rangle \in L^h_{\lambda,\tilde{\lambda}}$. This follows since $|\bar{S}'\rangle$ cannot be a null-state because then so would also $|\bar{S}\rangle$. Hence $|\bar{S}\rangle$ is BRST exact. $\square$

We thus see, as stated in the introduction, that the non-trivial BRST invariant states $|\bar{S}\rangle$ of negative ghost number come from the $\hat{Q}|\bar{S}\rangle = |\bar{N}\rangle$ part of the Kugo-Ojima quartet. In order to see how the states evolve from the Kugo-Ojima quartet we now introduce a trick due to Jantzen [5]. Let $z=\sum_i z^i\mu^i$ where the sum runs over all fundamental weights $\mu^i$ and $z^i \neq 0$ $\forall i$. Take highest weights $\lambda$, $\tilde{\lambda}$, and introduce a perturbation $\epsilon$ such that $\lambda \to \lambda+\epsilon z$ and $\tilde{\lambda} \to \tilde{\lambda}-\epsilon z$. Define the module $M^h_{\epsilon,\lambda,\tilde{\lambda}} \equiv M^h_{\lambda+\epsilon z} \times \tilde{M}^h_{\tilde{\lambda}-\epsilon z}$. For $0<\epsilon\ll 1$ it follows from the Kac-Kazhdan determinant [12] that $M^h_{\epsilon,\lambda,\tilde{\lambda}}$ is irreducible. Then by corollary 1 $\hat{\mathrm{H}}^p(\hat{h}_k, \hat{h}_{\tilde{k}}; M^h_{\epsilon,\lambda,\tilde{\lambda}})$ is zero for $p \neq 0$, and only singlets and quartets exists in the Verma module. Consequently we see that if singlet pairs exist, they will appear from quartets in the limit $\epsilon \to 0$.

Using the perturbation $\epsilon$ we now look for quartets $|N\rangle_\epsilon, |\bar{N}\rangle_\epsilon, |S\rangle_\epsilon, |\bar{S}\rangle_\epsilon$, with the properties: Two states $|N\rangle_\epsilon$, $|\bar{N}\rangle_\epsilon$ are null-states as $\epsilon \to 0$ and two are not null in this limit. In addition, they couple as ${}_\epsilon\langle\bar{N}|N\rangle_\epsilon \neq 0$ for $\epsilon \neq 0$ and $\langle\bar{S}|S\rangle \neq 0$. For later reference we will use a terminology from Jantzen [5] of filtered Verma modules. We may decompose the Verma module as $M^h_{\epsilon,\lambda,\tilde{\lambda}} = M^{(0)}_\epsilon \supset M^{(1)}_\epsilon \supset \ldots$ where we define the submodules $M^{(i)}_\epsilon$ to contain all states $|S\rangle_\epsilon$ such that ${}_\epsilon\langle\bar{S}|S\rangle_\epsilon$ is divisible by $\epsilon^i$. In the limit $\epsilon \to 0$ this induces a filtration of modules $M^h_{\lambda,\tilde{\lambda}} = M^{(0)} \supset M^{(1)} \supset \ldots$ . The states in $M^{(1)}/M^{(2)}$ will be referred to as first generation null-states. We may also note that the irreducible state space $L^h_{\lambda,\tilde{\lambda}} \equiv L^h_\lambda \times \tilde{L}^h_{\tilde{\lambda}}$ is isomorphic to $M^{(0)}/M^{(1)}$.

In order to explicitly construct the singlet pairs, we now study which null-states will appear in the relevant quartets. For $p<0$ we have from corollary 2 that $\hat{Q}|\bar{S}\rangle = |\bar{N}\rangle$ which means that $\hat{Q}|\bar{N}\rangle = 0$ on $M^h_{\lambda,\tilde{\lambda}}$. Now if there exists a state $|\bar{N}'\rangle$ such that $|\bar{N}\rangle = \hat{Q}|\bar{N}'\rangle$ then $|\bar{S}\rangle$ is not in the relative cohomology. This follows since $\hat{Q}(|\bar{S}\rangle+|\bar{N}'\rangle) = 0$ implies by corollary 2 that $|\bar{S}\rangle+|\bar{N}\rangle$ is BRST exact. Consequently

$|\bar{S}\rangle$ is BRST exact on $L^h_{\lambda,\tilde{\lambda}}$, since here states are only defined modulo null-states. We thus find that $|\bar{N}\rangle$ must be in the relative cohomology on $M^{(1)}$. The non-triviality of the relative cohomology at ghost number $-q+1$ will, therefore, be essential for the non-triviality of the relative cohomology at $p=-q$. Conversely, given a null-state $|\bar{N}\rangle$ of negative ghost number which is in the relative cohomology on $M^{(1)}/M^{(2)}$, then by the theorem $|\bar{N}\rangle = \hat{Q}|\bar{S}\rangle$ for some $|\bar{S}\rangle$ which does not belong $M^{(1)}$. Hence, it must be in the irreducible module $L^h_{\lambda,\tilde{\lambda}}$ and thus the existence of $|\bar{N}\rangle \in M^{(1)}/M^{(2)}$ implies the existence of $|\bar{S}\rangle$. Furthermore, knowing the former explicitely will, through the use of the homotopy operator, make it possible to construct the latter.

For $p>0$ we have, for states $|N\rangle_\epsilon$ and $|S\rangle_\epsilon$ in a quartet for $\epsilon \neq 0$, that $\hat{Q}|N\rangle_\epsilon = f(\epsilon)|S\rangle_\epsilon$, and $lim_{\epsilon\to 0} f(\epsilon) = 0$ if $|S\rangle$ is in the cohomology. Note that $\hat{Q}|S\rangle_\epsilon = 0$ for all values of $\epsilon$. Since $\hat{Q}$ is linear in $J$ and $\tilde{J}$ we will in fact have $f(\epsilon) \propto \epsilon$. We thus look for null-states satisfying $\hat{Q}|N\rangle = 0$ for $\epsilon = 0$. These can be found as follows. Take an appropriate submodule $M^h_{\lambda',\tilde{\lambda}}$. Obviously all null-states of the form $|N\rangle = \hat{Q}|N'\rangle$ can be discarded. A possible choice is then to take $|N\rangle$ to belong to the cohomology $\hat{\mathrm{H}}^{p-1}(\hat{h}_k, \hat{h}_{\tilde{k}}; L_{\lambda',\tilde{\lambda}})$ where $L_{\lambda',\tilde{\lambda}}$ is contained in $M^{(1)}/M^{(2)}$. Then $\hat{Q}|N\rangle = 0$, and $\hat{Q}|N\rangle_\epsilon = \epsilon|S\rangle_\epsilon \in M^h_{\epsilon,\lambda,\tilde{\lambda}}$. A state $|S\rangle$ found in this way will then be a non-trivial state in the cohomology. This may be proven by a more detailed analysis [3]. Thus non-trivial states at ghost number $p$ may be obtained from non-trivial states at ghost number $p-1$. Another possibility is to exchange the rôle of the modules $M^h_\lambda$ and $\widetilde{M}^h_{\tilde{\lambda}}$, but this will give us cohomologically equivalent states to the ones above. This may be proven using the homotopy operator introduced in the proof of the theorem.

Consider ghost number zero. We pick a Kugo-Ojima singlet $|P_0\rangle_\epsilon = |0,\lambda + \epsilon z\rangle|0,\tilde{\lambda}-\epsilon z\rangle|0\rangle^+$ with $\lambda+\tilde{\lambda}+\rho = 0$. The null-states of the quartet are constructed by the substitution of the primary $|0,\lambda\rangle$ by a corresponding highest weight null-state $|n_0\rangle$ of first generation i.e. $|P_0\rangle_\epsilon \to |N_0\rangle_\epsilon$. $|S_1\rangle$ is then found by applying $\hat{Q}$, $\hat{Q}|N_0\rangle_\epsilon = \epsilon|S_1\rangle_\epsilon$, and taking the limit $\epsilon \to 0$. It is clear that $|S_1\rangle$ is of the form $|s_1\rangle|0,\tilde{\lambda}\rangle|gh\rangle$ where $|gh\rangle$ is the appropriate ghost state which includes only $c$ ghosts. If we instead make the substitution $|0,\tilde{\lambda}\rangle_\epsilon \to |\tilde{n}\rangle_\epsilon$, then we would get a cohomologically equivalent state.

The next step is to find the state $|\bar{N}_0\rangle$ which satisfies $\hat{Q}|\bar{N}_0\rangle = 0$ and couples to $|N_0\rangle_\epsilon$ for $\epsilon \neq 0$. Since ${}_\epsilon\langle N_0|N_0\rangle_\epsilon \sim \epsilon$ we can take $|\bar{N}_0\rangle = |N_0\rangle + \dots$ . In order to find the rest of $|\bar{N}_0\rangle$, we use the following trick. Start by taking $|N_0\rangle$. Since it satisfies $\hat{Q}|N_0\rangle = 0$, we can use the homotopy operator $\kappa_0$ introduced below the theorem, but with the original sector replacing the auxiliary sector. Then we will

get $|N_0\rangle = \hat{Q}|\bar{S}_{-1}\rangle - |\tilde{\bar{N}}_0\rangle$ for some state $|\bar{S}_{-1}\rangle$. We now take $|\bar{N}_0\rangle \equiv |N_0\rangle + |\tilde{\bar{N}}_0\rangle$ and by this we have found the state $|\bar{S}_{-1}\rangle$ at ghost number minus one.

For ghost number two we proceed exactly as above. Let $|S_1\rangle$ be a state in the cohomology determined as above. Introduce Jantzen's perturbation, and substitute the primary state with a highest weight null-state in the original sector $|S_1\rangle_\epsilon \rightarrow |N_1\rangle_\epsilon$. Act on this by the BRST operator which will give $\hat{Q}|N_1\rangle_\epsilon = \epsilon|S_2\rangle_\epsilon$, etc.

Ghost number minus two follows essentially the same steps as minus one although finding the BRST invariant null-state is more complicated. We must, in the Verma module $M^h_{\lambda,\tilde{\lambda}}$, find the appropriate null-states at ghost number minus one, which belongs to the cohomology at $p=-1$ of $M^{(1)}/M^{(2)}$. We take as a general ansatz $\sum_i a_i|\bar{N}_{-1}\rangle_i$ where $|\bar{N}_{-1}\rangle_i$ is constructed from states $|\bar{S}_{-1}\rangle_i$ in the cohomology for an appropriate weight, by substitution of a primary in $|\bar{S}_{-1}\rangle_i$ with a highest weight null-state in either the original or the auxiliary sector. The requirement $\hat{Q}\sum_i a_i|\bar{N}_{-1}\rangle_i = 0$ determines the coefficients $a_i$. We know from the theorem of Kugo-Ojima, that the number of solutions to this equation is exactly the same as the number of solutions at ghost number two. Given a solution to $\hat{Q}\sum_i a_i|\bar{N}_{-1}\rangle_i = 0$ we may iteratively use the homotopy operator $\kappa_0$ to find a representative in the cohomology at ghost number minus two.

In order to proceed to arbitrary ghost numbers we repeat this process the desired number of steps following the schematic pattern depicted below.

$$\begin{array}{l}
\ldots \xrightarrow{\hat{Q}} \epsilon|S_{p-1}\rangle_\epsilon \longrightarrow |N_{p-1}\rangle_\epsilon \xrightarrow{\hat{Q}} \epsilon|S_p\rangle_\epsilon \longrightarrow |N_p\rangle_\epsilon \xrightarrow{\hat{Q}} \epsilon|S_{p+1}\rangle_\epsilon \longrightarrow \ldots \\
\ldots \xrightarrow{\kappa} \hat{Q}|\bar{S}_{-p+1}\rangle \longrightarrow |\bar{N}_{-p+1}\rangle \xrightarrow{\kappa} \hat{Q}|\bar{S}_{-p}\rangle \longrightarrow |\bar{N}_{-p}\rangle \xrightarrow{\kappa} \hat{Q}|\bar{S}_{-p-1}\rangle \longrightarrow \ldots
\end{array}$$

where $\kappa$ is the extension of $\kappa_0$ to all orders.

For the explicit construction we must know the explicit form of highest weight null-states in the Verma module, which are given in [4]. Let us outline the construction in [4] of highest weight null-states. First we observe that the affine weight lattice for $k > -c_h$ may be constructed from the set of dominant weights and the Weyl group. Let $\hat{\mu}_0 + \hat{\rho}/2$ be a dominant weight, i.e. $(\hat{\mu}_0 + \hat{\rho}/2)\cdot\hat{\alpha}^i \geq 0$ for all simple affine roots $\hat{\alpha}^i$. $\hat{\rho}$ is defined by $\hat{\rho}\cdot\hat{\alpha}^i = \hat{\alpha}^i\cdot\hat{\alpha}^i$. Take an affine weight $\hat{\lambda}$. Then there exists a sequence of $\hat{\rho}$-centered simple Weyl reflexions, $\sigma^\rho_i(\mu_0) \equiv \sigma_i(\hat{\mu}_0 + \hat{\rho}/2) - \hat{\rho}/2$, such that $\sigma^\rho_{i_p}\sigma^\rho_{i_{p-1}}\ldots\sigma^\rho_{i_1}(\hat{\mu}_0) < \sigma^\rho_{i_{p-1}}\ldots\sigma^\rho_{i_1}(\hat{\mu}_0)$, and $\hat{\lambda} = \sigma^\rho_{i_p}\ldots\sigma^\rho_{i_1}(\hat{\mu}_0)$. Furthermore, by requiring the "word" $(i_1,\ldots,i_p)$ to be of minimum length, $\hat{\mu}_0$ and this word are uniquely determined by $\hat{\lambda}$. Hence, we may use the set of Weyl reflexions and the

dominant weights to parametrize any weight $\hat{\lambda}$ for which $k > -c_h$. For the auxiliary sector, which has $\tilde{k} < -c_h$, one may proceed similarly for $-\hat{\tilde{\lambda}} - \hat{\rho}$. For later reference we define in this parametrization, the length $l_\lambda$ of a weight $\hat{\lambda}$, to be the number of entries in the word. For the auxiliary sector the length $l_{\tilde{\lambda}}$ of a weight $\hat{\tilde{\lambda}}$ is similarly defined using the weight $-\hat{\tilde{\lambda}} - \hat{\rho}$. The $\hat{\rho}$-centered Weyl reflexion $\sigma_i^\rho$ of $\hat{\mu}_0$ may be represented at the level of states as $|\mu_0\rangle \to (f_i)^{\gamma_i}|\mu_0\rangle$, where $f_i$ is the affine generator corresponding to $-\hat{\alpha}_i$, and $\gamma_i$ is defined from $\sigma_i^\rho(\hat{\mu}_0 + \hat{\rho}/2) - \hat{\rho}/2 - \hat{\mu}_0 \equiv -\gamma_i \hat{\alpha}_i$. It is straightforward to verify that the state $(f_i)^{\gamma_i}|\mu_0\rangle$ is a highest weight null-state. The procedure may be repeated to produce two different highest weight null-states $|\lambda_{i_1 \dots i_n}\rangle = (f_{i_n})^{\gamma_{i_n}} \dots (f_{i_1})^{\gamma_{i_1}}|\mu_0\rangle$ and $|\lambda_{j_1 \dots j_m}\rangle = (f_{j_m})^{\gamma_{j_m}} \dots (f_{j_1})^{\gamma_{j_1}}|\mu_0\rangle$. By eliminating $|\mu_0\rangle$ we may formally write

$$|n\rangle \equiv |\lambda_{i_1 \dots i_n}\rangle = (f_{i_n})^{\gamma_{i_n}} \dots (f_{i_1})^{\gamma_{i_1}} (f_{j_1})^{-\gamma_{j_1}} \dots (f_{j_m})^{-\gamma_{j_m}} |\lambda_{j_1 \dots j_m}\rangle \qquad (2)$$

$|n\rangle$ may not exist as a state in the Verma module over $|\lambda_{j_1 \dots j_m}\rangle$ due to negative powers appearing on the right hand side. It may, however, be shown [4] that if the word of $j_1 \dots j_m$ may be obtained from $i_1 \dots i_n$ by deleting $n - m$ letters of the latter word, then $|n\rangle$ exists. The right hand side may then be put in a well-defined form, where only positive powers of the generators appear, and $|n\rangle$ is the required highest weight null-state. If $m = n - 1$ then the null-states are of first generation. We also note that if $|N\rangle$ is a first generation highest weight null-state of weight $\lambda', \tilde{\lambda}'$ in $M^h_{\lambda,\tilde{\lambda}}$ then $l_{\lambda'} + l_{\tilde{\lambda}'} - l_\lambda - l_{\tilde{\lambda}} = 1$.

Let us now discuss what ghost numbers appear for which representations. At ghost number zero we are constrained by $\lambda_0 + \tilde{\lambda}_0 + \rho = 0$, which means that $l_{\lambda_0} - l_{\tilde{\lambda}_0} = 0$. We now recall that for ghost number one we used the substitution $|0, \lambda_0\rangle|0, \tilde{\lambda}_0\rangle \to |n_0\rangle|0, \tilde{\lambda}_0\rangle$ where $|n_0\rangle \in M^{(1)}/M^{(2)}$. This means that $l_{\lambda_1} - l_{\lambda_0} = 1$ where $\lambda_1$ is the weight of the state at ghost number one produced as above from $|n_0\rangle_\epsilon|0, \tilde{\lambda}_0 - \epsilon t\rangle|0\rangle^+$. We thus have that $l_{\lambda_1} - l_{\tilde{\lambda}_0} = 1$. The argument may be repeated, and one realizes that for the generic case the ghost numbers are given by $p = \pm|l_\lambda - l_{\tilde{\lambda}}|$. Here the absolute value arises from the fact that one may use the substitution of primary to null-state in the auxiliary sector as well. It is thus possible to fix the representation of say the original sector and adjust the auxiliary sector to obtain any ghost number. For the special case of integrable representations of $\hat{h}$ we have $l_\lambda = 0$ and consequently the ghost numbers are given by $p = \pm l_{\tilde{\lambda}}$.

We have this far performed our analysis on the module $L^h \times \tilde{L}^h$. If we embed $L^h$ into $L^g$ for a general algebra $\hat{g}$, of which $\hat{h}$ is a subalgebra, our results change

in the following way. The theorem as well as the two corollaries remain true. This follows since in proving these, we only used the auxiliary sector for which state space is unchanged. Thus it is still necessary for non-trivial states with negative ghost number to satisfy $\hat{Q}|\bar{S}\rangle = |\bar{N}\rangle$ for some null-state $|\bar{N}\rangle$. This null-state is always a sum of null-states in the auxiliary as well as original sectors. Similarly for positive ghost numbers we have the equation $\hat{Q}|N\rangle_\epsilon = \epsilon|S\rangle_\epsilon$. This shows that any solution of the H/H case will be contained in the G/H case, provided $|\bar{N}\rangle$ is also a null-vector w.r.t. $\hat{g}$.

Now, if the irreducible $\hat{g}$ module does not completely decompose into a sum of irreducible $\hat{h}$ modules, then this implies the existence of at least one highest weight null-state w.r.t. $\hat{h}$ which is not a null-state w.r.t. $\hat{g}$ (see [2]). If this in turn implies that $|\bar{N}\rangle$ is not null, then $|\bar{S}\rangle$ is not BRST invariant and consequently will no longer be in the cohomology. Similarly for positive ghost numbers, we have states $|S\rangle$ satisfying $\hat{Q}|N\rangle_\epsilon = \epsilon|S\rangle_\epsilon$. This relation will still be true, even in the case when the conjugate state to $|N\rangle$ is no longer a null-state. However, then $|S\rangle$ must be BRST exact. In our construction we have shown how non-trivial states at ghost number $\pm p$ generate new non-trivial states at $\pm(p+1), \pm(p+2), \ldots$ . If a particular singlet pair will drop out of the cohomology, it may therefore imply that many, possibly infinitely many, states will drop out of the cohomology. If this is actually the case is still an open question.

For the special case of integrable representations of the original sector, it has been proved [13] that the irreducible $\hat{g}$ module decomposes completely into a sum of irreducible $\hat{h}$ modules. Then it is easy to show that every null-state of $\hat{h}$ is also a null-state of $\hat{g}$, from which it follows that the analysis and construction given in this paper apply without restrictions.

Let us end by presenting some explicit examples for the choice of $\hat{h} = \widehat{su}(2)$ with affine generators $J_n^+, J_n^-$ and $J_n^3$ . We will content ourselves with giving a few states, and only for ghost numbers $\pm 1, \pm 2$.

We start with the primary satisfying $j_0 + \tilde{j}_0 + 1 = 0$: $|j_0 = -\frac{k}{2} - 2\rangle|\tilde{j}_0 = \frac{k}{2} + 1\rangle|0\rangle^+$. We intend to use the primary to highest weight null-state substitution $|j_0\rangle \to (J_0^-)^{k+3}|j_1 = \frac{k}{2}+1\rangle$. Following our procedure we introduce the parameter $\epsilon$ and take $j_1 \to j_1 + \epsilon$ and $\tilde{j}_0 \to \tilde{j}_0 - \epsilon$ in order to preserve the zero eigenvalue of $J_0^{3,tot}$. Acting on this state with $\hat{Q}$ will, in the limit $\epsilon \to 0$, give us the non-trivial BRST invariant state $c_0^-(J_0^-)^{k+2}|j_1\rangle|\tilde{j}_0\rangle|0\rangle^+$. For ghost number minus one we use the homotopy operator

on the highest weight null-state to find $b_0^- \sum_{i=0}^{k+2}(-1)^i(J_0^-)^{k+2-i}(\tilde{J}_0^-)^i|j_1\rangle|\tilde{j}_0\rangle|0\rangle^+$ in the cohomology.

In the second step we substitute $|j_1 = \frac{k}{2}+1\rangle \to J_{-1}^+|j_2 = \frac{k}{2}\rangle$ in the BRST invariant state at ghost number one. After the usual steps of applying $\hat{Q}$ and taking $\epsilon$ to zero, we find the resulting state $c_0^- c_{-1}^-(J_0^-)^k|j_2\rangle|\tilde{j}_0\rangle|0\rangle^+$. We have here dropped several terms which in the limit $\epsilon \to 0$ became null-states. By inspection it is obvious in this example that there exists only one state at ghost number two for these representations, and hence the cohomology is here one-dimensional.

For ghost number minus two we will only give the null-state $|\bar{N}_{-1}\rangle$ of ghost number minus one, which is BRST exact and from which one may construct the state at ghost number minus two. Following our general discussion above, we will use states at ghost number minus one, and rewrite them as null-states. To construct the former, we introduce the null-states $n_1|j_1\rangle \equiv (J_{-1}^+(J_0^-)^2 - (k+3)J_{-1}^3 J_0^- - \frac{(k+2)(k+3)}{2}J_{-1}^-)|j_1 = -\frac{k}{2}-1\rangle$, $n_2|j_2\rangle \equiv (J_0^-)^{k+1}|j_2 = \frac{k}{2}\rangle$, $n_3|j_3\rangle \equiv (J_0^-)^{k+3}|j_3 = \frac{k}{2}+1\rangle$, $n_4|j_2\rangle \equiv J_{-1}^+|j_2\rangle$. In the auxiliary sector we get $\tilde{n}_1|\tilde{j}_1\rangle$ and $\tilde{n}_4|\tilde{j}_4\rangle$ from $n_1|j_1\rangle$ and $n_4|j_4\rangle$ by the substitution $J \to \tilde{J}$ and $k \to -k-4$. Also $\tilde{n}_2|\tilde{j}_2\rangle$ and $\tilde{n}_3|\tilde{j}_1\rangle$ follow from $n_2|j_2\rangle$ and $n_3|j_3\rangle$ by substitution $J \to \tilde{J}$. The relevant states $s_1|j_1\rangle|\tilde{j}_1\rangle|0\rangle^+, s_3|j_3\rangle|\tilde{j}_1\rangle|0\rangle^+, \tilde{s}_2|j_2\rangle|\tilde{j}_2\rangle|0\rangle^+, \tilde{s}_4|j_2\rangle|\tilde{j}_4\rangle|0\rangle^+$ for ghost number minus one may be found, using the homotopy operator, from the null-states satisfying the equations $\hat{Q}s_1|j_1\rangle|\tilde{j}_1\rangle|0\rangle^+ = (n_1+\tilde{n}_1)|j_1\rangle|\tilde{j}_1\rangle|0\rangle^+$, $\hat{Q}s_3|j_3\rangle|\tilde{j}_1\rangle|0\rangle^+ = (n_3 + (-1)^{k+2}\tilde{n}_3)|j_3\rangle|\tilde{j}_1\rangle|0\rangle^+$, $\hat{Q}\tilde{s}_2|j_2\rangle|\tilde{j}_2\rangle|0\rangle^+ = (n_2 + (-1)^k\tilde{n}_2)|j_2\rangle|\tilde{j}_2\rangle|0\rangle^+$, and $\hat{Q}\tilde{s}_4|j_2\rangle |\tilde{j}_4\rangle|0\rangle^+ = (n_4+\tilde{n}_4)|j_2\rangle|\tilde{j}_4\rangle|0\rangle^+$. We now use primary to null-state substitutions $|j_1\rangle \to n_2|j_2\rangle$, $|j_3\rangle \to n_4|j_2\rangle$, $|\tilde{j}_2\rangle \to \tilde{n}_1|\tilde{j}_1\rangle$, and $|\tilde{j}_4\rangle \to \tilde{n}_3|\tilde{j}_1\rangle$, and demand that $\hat{Q}|\bar{N}_{-1}\rangle = \hat{Q}(a_1 s_1 n_2 + a_3 s_3 n_4 + a_2\tilde{s}_2\tilde{n}_1 + a_4\tilde{s}_4\tilde{n}_3)|j_2\rangle|\tilde{j}_1\rangle|0\rangle^+ = 0$, where $a_i$ are constants. This is satisfied for $a_3 = -a_1$, $a_2 = -a_1$, and $a_4 = (-1)^{k+2}a_1$. We have here used $n_1 n_2 = n_3 n_4$, and $\tilde{n}_2\tilde{n}_1 = \tilde{n}_4\tilde{n}_3$, which may be calculated explicitly or understood from the uniqueness of highest weight null-states. All that remains is to use the homotopy operator to construct $|\bar{S}_{-2}\rangle$ from the equation $|\bar{N}_{-1}\rangle = \hat{Q}|\bar{S}_{-2}\rangle$.

## References


[1] D. Karabali, and H. Schnitzer, *BRST quantization of the gauged WZW action and coset conformal field theories*, Nucl. Phys. B329 (1990) 649

[2] S. Hwang and H. Rhedin, *The BRST formulation of G/H WZNW models*, Nucl. Phys. B406 (1993) 165

[3] S. Hwang, work in preparation.

[4] F.G. Malikov, B.L. Feigin and D.B. Fuks, *Singular vectors in Verma modules over Kac-Moody algebras*, Funkt. Anal. Ego Prilozh. 20 (1986) 25

[5] J.C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture notes in Mathematics, Eds. A. Dold and B. Eckmann, Springer-Verlag, 1979

[6] T. Kugo and I. Ojima, *Local covariant operator formalism of non-abelian gauge theories and quark confinement problem*, Suppl. Prog. Theor. Phys. 66 (1979) 1

[7] P. Bouwknegt, J. McCarthy, and K. Pilch *Semi-infinite cohomology in conformal field theory, and 2d gravity.*, CERN-TH.6646/92

[8] O. Aharony, O. Ganor, J. Sonnenschein, S. Yankielowicz, and N. Sochen, *Physical states in* $\frac{G}{G}$ *models, and two-dimensional gravity*, Nucl. Phys. B399 (1993) 527
O. Aharony, J. Sonnenschein, and S. Yankielowicz, *G/G models and* $W_N$ *strings*, Phys. Lett. B289 (1992) 309
O. Aharony, O. Ganor, J. Sonnenschein, and S. Yankielowicz, *On the twisted* $\frac{G}{H}$ *toplogical models*, Nucl. Phys. B399 (1993) 560

[9] H.-L. Hu and M. Yu, *On BRST cohomology of* $SL(2,R)_{p/q-2}/SL(2,R)_{p/q-2}$ *gauged WZNW models*, Nucl. Phys. B391 (1993) 389

[10] K. Gawędski and A. Kupiainen, *Coset construction from functional integrals*, Nucl. Phys. B320 (1989) 625

[11] D. Karabali, Q-H. Park, H. Schnitzer, and Z. Yang, *A GKO coset construction based on a path integral formulation of gauged Wess-Zumino-Witten actions*, Phys. Lett. B216 (1989) 307

[12] V.G. Kac, and D.A. Kazhdan, *Structure of representations with highst weight of infinite-dimensional Lie algebras*, Adv. in Math. 34 (1979) 97

[13] V.G. Kac, and D. Peterson, *Infinite dimensional Lie algebras, theta functions and modular forms*, Adv. Math. 53, (1984) 125